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On the deep water wave motion

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Abstract

The problem of the propagation of surface waves over deep water is considered. We present a rigorous approach towards the only known (non-trivial) explicit solution to the governing equations for water waves—Gerstner's wave. Some properties of this solution, and how these relate to some basic conclusions about water waves that may be observed experimentally, are discussed.

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1. Introduction

When one watches the sea from the shore it is possible to trace a wave as it comes in from the open sea. It seems that at every instant one is looking at a quite different piece of water. This is not so: what one can observe travelling across the sea to the shore is not the water, it is a shape, a pattern that may become more marked or may fade out and disappear. It has been well confirmed by experiment that, as a surface wave passes over deep water, the surface water particles trace roughly circular orbits with a diameter equal to the height of the wave (see figure 1). As the wave crest approaches, the surface water particles rise with respect to the still-water level and move forward, while when the crest passes, the particles begin to fall. With the advancement of the trough, the particles slow their falling rate and move backward, while at the bottom of the trough they move only backward. As the trough passes, the particles start to rise again. Below the surface the diameter decreases and orbital motion practically ceases if the depth exceeds half a wavelength (see figure 2).

It is this orbital motion of the surface water particles that causes a floating object to move up and down, forward and backward, as the waves pass under it. The fact that the wave motion

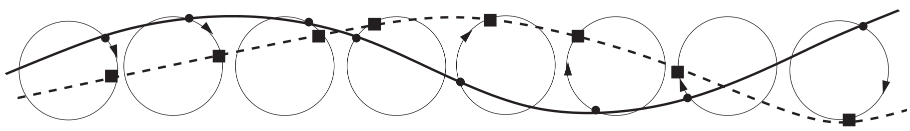

Figure 1. The orbits of surface particles in a deep water wave. The particles move from ● to ■ as the wave moves from the full curve to the broken curve.

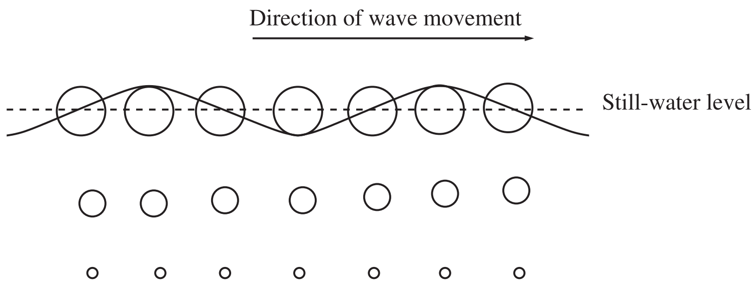

Figure 2. Clockwise orbital motion in circles (deep water).

fades out with increasing depth is used by submarines that dive deeply during storms in the open sea. Also, designers of floating structures such as semi-submersible rigs used for oil drilling set the buoyant parts at great depth, so that the structure is not appreciably affected by surface waves.

A suitable description of deep water waves is obtained by assuming the water to be (theoretically) infinitely deep, the water body being the region of $\mathbb{R}^3$ bounded above by the free surface $z = h(t, x, y)$. Let $\boldsymbol{u} = (u, v, w)$ be the velocity field and let us recall the general problem for the propagation of gravity waves in deep water cf [3]. Homogeneity (constant density ρ) is a good approximation for water, cf [2], so that we have the equation of mass conservation in the form[1]

$$u_x + v_y + w_z = 0. \tag{1.1}$$

On the other hand, under the assumption that water is inviscid, the equation of motion is Euler's equation (see [5])

$$\begin{aligned} \frac{\mathrm{D}u}{\mathrm{D}t} &= -\frac{1}{\rho} P_x \\ \frac{\mathrm{D}v}{\mathrm{D}t} &= -\frac{1}{\rho} P_y \\ \frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{1}{\rho} P_z - g. \end{aligned} \tag{1.2}$$

where $P(t, x, y, z)$ denotes the pressure, g is the gravitational acceleration constant and $\mathrm{D}/\mathrm{D}t$ is the material time derivative, $\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$, expressing the rate of change of the quantity f associated with the same fluid particle as it moves about. The continuity equation (1.1) and Euler's equation (1.2) are the exact equations of motion for the velocity field. Let us now present the boundary conditions which select the water-wave problem from all other possible solutions of equations (1.1) and (1.2). The dynamic boundary condition

$$P = P_0 \qquad \text{on} \quad z = h(t, x, y) \tag{1.3}$$

where P_0 is the constant atmospheric pressure, decouples the motion of the air from that of the water. The kinematic boundary condition is

$$w = h_t + uh_x + vh_y \qquad \text{on} \quad z = h(t, x, y) \tag{1.4}$$

expressing the fact that the same particles always form the free water surface, i.e. $\frac{\mathrm{D}}{\mathrm{D}t}(z-h) = 0$. The boundary condition at the bottom

$$\boldsymbol{u} \to 0 \qquad \text{as} \quad z \to -\infty \tag{1.5}$$

[1] We recognize in (1.1) the condition of incompressibility: in general, a fluid moving with velocity $\boldsymbol{u}$ is being compressed at the rate $\nabla \cdot \boldsymbol{u}$ and for this reason our velocity field is divergence free.

expresses the fact that at great depths there is practically no motion. The general description of the propagation of a wave over deep water is encompassed by equations (1.1)–(1.5). A distinctive feature is that the free surface is not known and must be determined as part of the solution.

There is no rigorous theory so far yielding an explanation of the aspects of water waves that have been uncovered within the context of approximate theories but there has been significant recent progress [1, 6, 8]. Most basic results on the dynamics of water waves have been obtained by perturbation schemes used to replace the governing equations by approximate models that have been used extensively to make a variety of theoretical studies, which have been confirmed in experimental contexts. The rigorous justification of these approximations is conceptually and technically difficult, but the lack of deductive rigour can lead to the fallability of the physically plausible arguments. In particular, the existing linear approximate theory (see [3]) explaining the described motion of the deep water particles as a wave passes is still essentially tentative. The same linear approximation shows that the propagation velocity of a deep water wave depends on the wavelength—deep water waves are dispersive.

This paper is devoted to the only known explicit solution to the governing equations (1.1)–(1.5), whose existence was pointed out by Gerstner in 1802 (cf [4]). Using an interplay of topological and analytical ideas we show rigorously that Gerstner's calculations can be justified in a satisfactory mathematical manner. To the best of our knowledge, a rigorous analysis of Gerstner's wave has not previously been given. The final section gives a brief description of how Gerstner's wave reflects the conclusions regarding the dispersive character of the waves and the motion of the particles in deep water, providing a background against which the predictions of the approximate linear theory can be checked. We would like to point out that Gerstner's wave does not belong to the class of Stokes waves (see [6] for a self-contained analysis of the rigorous theory for this type of waves) as the water motion induced by it is rotational, in contrast to the irrotational character of a Stokes wave—note also that no representation of a Stokes wave by a mathematical expression of closed form is known cf [6].

2. Gerstner's wave

Let us first present the main ideas behind Gerstner's construction. Gerstner's wave is a two-dimensional wave—the motion is identical in all planes parallel to a representative fixed vertical plane (a particle in such a plane will stay forever in it). It is therefore sufficient to consider the motion of the water particles in the fixed plane, say the (x, z)-plane, with velocity $\boldsymbol{u}$, pressure P and free surface h all not dependent on the y-variable. The analysis is considerably more transparent if we adopt the Lagrangian point of view by following the evolution of individual water particles.

Let a and b be parameters which fix the position of a particular water particle before the passage of a wave—with $a \in \mathbb{R}$ and $b \leqslant b_0$ for some $b_0 \leqslant 0$ fixed[2] the lower half-plane represents the still water body. Gerstner's wave is obtained by supposing that the position of this particular particle at time t is given by

$$\begin{aligned} x &= a + \frac{\mathrm{e}^{mb}}{m} \sin m\left(a + \sqrt{\frac{g}{m}}\, t\right) \\ z &= b - \frac{\mathrm{e}^{mb}}{m} \cos m\left(a + \sqrt{\frac{g}{m}}\, t\right) \end{aligned} \tag{2.1}$$

where $m > 0$ is fixed. If the still water surface was $\{(a, b_0) : a \in \mathbb{R}\}$ with $b_0 < 0$, then

[2] As we will see, the restriction to negative values is essential.

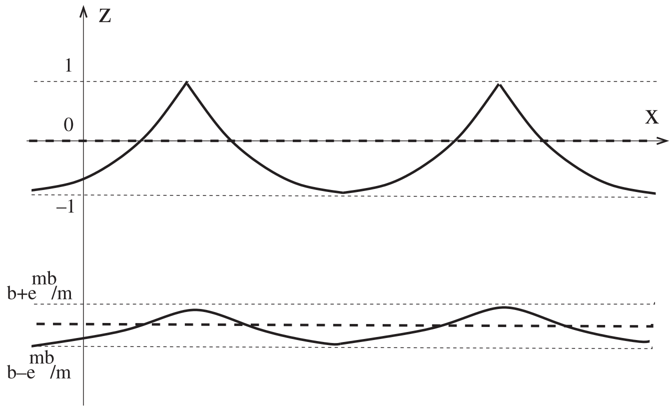

Figure 3. Profiles of the surface wave (thick full curve) and the original still water surface (thick dotted line).

the profile of the surface wave would be a smooth curve obtained by setting $b = b_0$ in (2.1); this curve is called a trochoid. The extreme case of the still water surface at $\{(a, 0) : a \in \mathbb{R}\}$ leads to a surface wave having the profile of a cycloid—a continuous curve with upward cusps (see figure 3). We would like to point out that the still water surface is used just as a convenient reference—Gerstner's wave is not developing from this state of the water body (otherwise, cf [5], the flow would be irrotational and we shall see that actually its vorticity is non-zero).

Relation (2.1) shows that the presumed path of a particle is the circle centred at (a, b) and whose radius is e^{mb}/m, the angular velocity with which the particle moves counterclockwise being $\sqrt{g/m}$. If we consider the motion of another particle we merely change the values of a and b in (2.1). It is implied that the creation of the wave produces a drift (moving the particle from the centre of the circle to a location on the circumference of the circle (see figure 4)) but, once the wave is created, as an effect of its passage the particle will move clockwise in a circular orbit—the direction of propagation of the surface wave being (in the adopted reference system) from right to left.

In order to show that (2.1) provides a solution to the governing equations, we have to prove that it defines a motion of the whole fluid body for which the equation of continuity

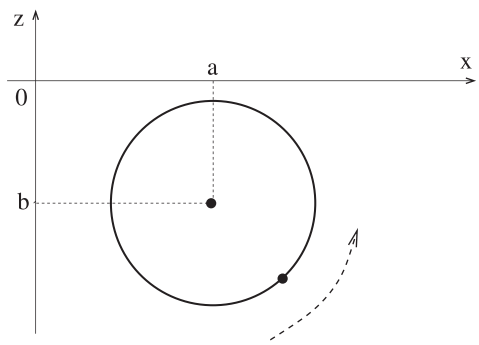

Figure 4. Path of a particle located in still water at the centre.

(1.1) holds, that a suitable pressure verifies Euler's equation and the boundary condition (1.3), a last point being that a particle on the free surface stays on the free surface—this would take care of the boundary condition (1.4)—and that the condition (1.5) at the bottom is verified. The delicate point is to show that it is possible to have a motion of the whole fluid body where all particles describe circles with a depth-dependant radius. For the other issues we mainly clarify and expand upon the discussion in [4].

Step 1. *The map* (2.1) *defines for* $t \geqslant 0$ *a motion of the water body located while still below the horizontal line* $\{(a, b_0) : a \in \mathbb{R}\}$ *for some* $b_0 \leqslant 0$. *That is, for every fixed* $t \geqslant 0$, *the map* (2.1) *defines a diffeomorphism from the domain below* $\{(a, b_0) : a \in \mathbb{R}\}$ *in the* (x, z)*-plane to the domain below the surface wave* $z = h(t, x)$. *The profile of the surface wave is smooth if* $b_0 < 0$, *while if* $b_0 = 0$ *we have a continuous, piecewise smooth curve with upward cusps.*

First of all, we may assume that $t = 0$ since the general case can be deduced from this particular situation by conjugation: changing variables $(a, b) \mapsto (a + t\sqrt{g/m}, b)$, performing the transformation (2.1) with $t = 0$, and then simply shifting the horizontal variable by an amount $t\sqrt{g/m}$. That is, it suffices to analyse the map

$$\begin{aligned} x &= a + \frac{\mathrm{e}^{mb}}{m} \sin ma \\ z &= b - \frac{\mathrm{e}^{mb}}{m} \cos ma. \end{aligned} \tag{2.2}$$

Keeping b fixed, the values of z in (2.2) recur when a is increased by $2\pi/m$, while the values of x undergo a linear shift of amount $2\pi/m$. For this reason, it is enough to restrict our attention to $a \in [0, 2\pi/m]$.

Let us start by analysing the image under (2.2) of the horizontal line $\{(a, b) : a \in \mathbb{R}\}$ with $b \leqslant b_0$. We visualize the image as the graph of a function expressing the horizontal coordinate in terms of the vertical coordinate: from the second component of (2.2) we find a as a function of b and z. For fixed b and $a \in [0, \pi/m]$ we obtain

$$\begin{aligned} &z \mapsto \frac{1}{m} \arccos\left[\frac{m(b-z)}{\mathrm{e}^{mb}}\right] + \frac{\mathrm{e}^{mb}}{m}\sqrt{1 - \frac{m^2}{\mathrm{e}^{2mb}}(b-z)^2} \\ &z \in \left[b - \frac{\mathrm{e}^{mb}}{m}, b + \frac{\mathrm{e}^{mb}}{m}\right] \end{aligned}$$

while for $a \in [\pi/m, 2\pi/m]$ we obtain

$$\begin{aligned} &z \mapsto \frac{2\pi}{m} - \frac{1}{m} \arccos\left[\frac{m(b-z)}{\mathrm{e}^{mb}}\right] - \frac{\mathrm{e}^{mb}}{m}\sqrt{1 - \frac{m^2}{\mathrm{e}^{2mb}}(b-z)^2} \\ &z \in \left[b - \frac{\mathrm{e}^{mb}}{m}, b + \frac{\mathrm{e}^{mb}}{m}\right]. \end{aligned}$$

We see now that if $b < 0$, then the image of the line $z = b$ is a smooth curve, with the z-values recurring when a is increased by $2\pi/m$. The profile of the restriction of this curve to $a \in [0, 2\pi/m]$ is rising on $[0, \pi/m]$ from $(0, b - \mathrm{e}^{mb}/m)$ to reach its peak $(\pi/m, b + \mathrm{e}^{mb}/m)$ and then it is descending for $a \in [\pi/m, 2\pi/m]$ towards its lowest point at $(2\pi/m, b - \mathrm{e}^{mb}/m)$ (see figure 5).

If $b = 0$, the situation is changing, as the two restrictions, to the intervals $[0, \pi/m]$ and $[\pi/m, 2\pi/m]$ respectively, have the derivative zero for $z \to 1$. In this case the image of the line $\{(a, 0) : a \in \mathbb{R}\}$ is a continuous, piecewise smooth curve with sharp peaks at the points $((2k + 1)\pi/m, 1)$, $k \in \mathbb{Z}$.

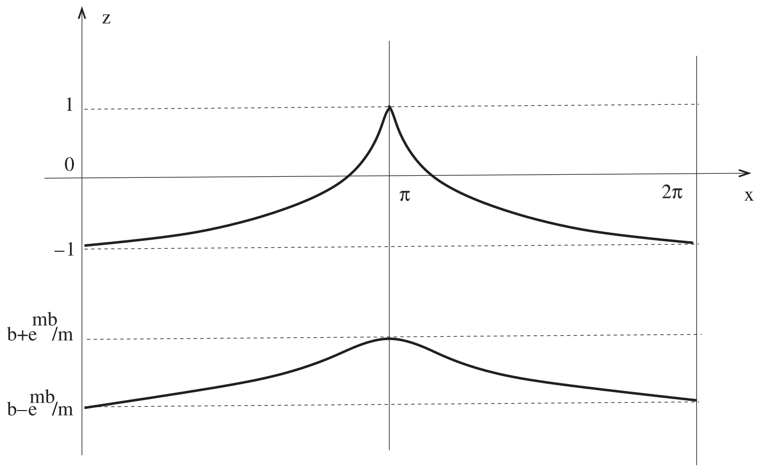

Figure 5.

It is useful to note that for $k \in \mathbb{Z}$, the vertical half-lines $\{(2k\pi/m, b) \,:\, b \leqslant b_0\}$ and $\{((2k+1)\pi/m, b) \,:\, b \leqslant b_0\}$ are mapped by (2.2) into $\{(2k\pi/m, z) \,:\, z \leqslant b_0 - e^{mb_0}/m\}$, respectively $\{((2k+1)\pi/m, z) \,:\, z \leqslant b_0 + e^{mb_0}/m\}$. This is a consequence of the fact that both functions $s \mapsto s - e^{ms}/m$, $s \mapsto s + e^{ms}/m$ are increasing on $(-\infty, 0]$.

The differential of (2.2) at a fixed point (a, b) with $b < 0$ is given by the 2×2 matrix

$$\begin{pmatrix} 1 + e^{mb}\cos ma & e^{mb}\sin ma \\ e^{mb}\sin ma & 1 - e^{mb}\cos ma \end{pmatrix}$$

of determinant $1 - e^{2mb} > 0$, so that from the inverse function theorem we infer that (2.2) is a local diffeomorphism from $\mathcal{U} = \{(a, b) \,:\, b < 0\}$ onto its image $\mathcal{W}$. We want to prove that $\mathcal{W}$ is precisely the part of the (x, z)-plane below the cycloid and that (2.2) is a global diffeomorphism from $\mathcal{U}$ to $\mathcal{W}$.

To prove that (2.2) is injective on $\mathcal{U}$ it is convenient to introduce $\xi = (x, z) \in \mathbb{R}^2$ and rewrite (2.2) in the form of an application $F(\xi) = \xi + f(\xi)$. If $|\cdot|$ is the Eulidean norm of $\mathbb{R}^2$, $|(x, z)| = \sqrt{x^2 + z^2}$, we have that

$$\begin{aligned} |F(\xi_2) - F(\xi_1)| &\geqslant |\xi_2 - \xi_1| - |f(\xi_2) - f(\xi_1)| \\ &\geqslant |\xi_2 - \xi_1| - \max_{s\in[0,1]}\{\|Df_{s\xi_1+(1-s)\xi_2}\|\}\,|\xi_2 - \xi_1| \end{aligned}$$

by the mean-value theorem. However,

$$Df_{(a,b)} = e^{mb}\begin{pmatrix} \cos ma & \sin ma \\ \sin ma & -\cos ma \end{pmatrix}$$

so that

$$\|Df_{(a,b)}\| = \max_{\theta\in[0,2\pi]} |Df_{(a,b)}(\cos\theta, \sin\theta)| = \max_{\theta\in[0,2\pi]} e^{mb}\big|\big(\cos(ma - \theta), \sin(ma - \theta)\big)\big| = e^{mb}$$

and therefore

$$|F(\xi_2) - F(\xi_1)| \geqslant (1 - e^{mb})|\xi_2 - \xi_1| \qquad b = \max\{b_1, b_2\}$$

for all $\xi_1 = (a_1, b_1)$, $\xi_2 = (a_2, b_2)$ in $\mathcal{U}$. The fact that F is one-to-one on $\mathcal{U}$ is now plain.

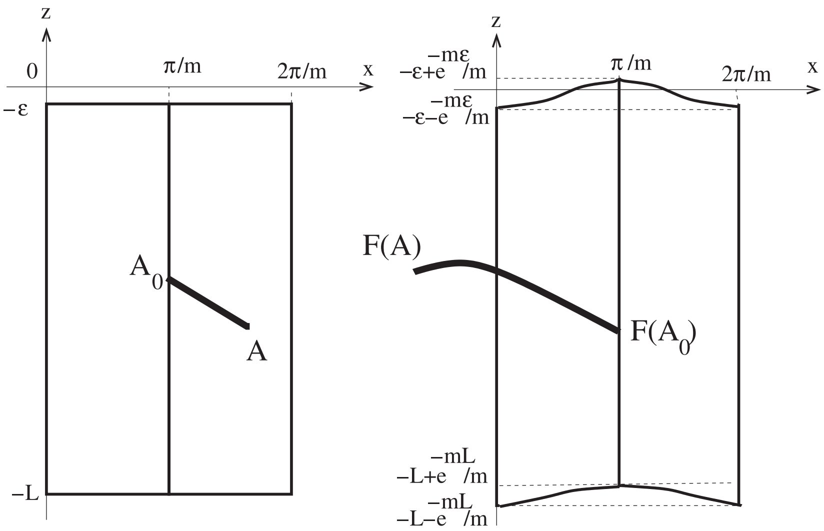

Figure 6.

Let us now show that $\mathcal{W}$ is the domain below the cycloid. Since $F(a + 2\pi/m, b) = F(a, b) + (2\pi/m, 0)$ it is enough to show that the intersection of $\mathcal{W}$ with the strip $0 \leqslant x \leqslant 2\pi/m$ is the part of the strip which lies below the cycloid. Let $\varepsilon > 0$, $L > 0$ and consider the image $\mathcal{W}_\varepsilon^L$ under F of the set $\mathcal{U}_\varepsilon^L = \{(a, b) : a \in [0, 2\pi/m], -L < b < -\varepsilon\}$. The image of the boundary $\partial\mathcal{U}_\varepsilon^L$ is known from our previous considerations. It consists of the two vertical segments $\{(0, z) : -L - \mathrm{e}^{-mL}/m < z < -\varepsilon - \mathrm{e}^{-m\varepsilon}/m\}$ and $\{(2\pi/m, z) : -L - \mathrm{e}^{-mL}/m < z < -\varepsilon - \mathrm{e}^{-m\varepsilon}/m\}$, together with the restriction to the strip $a \in [0, 2\pi/m]$ of the two trochoids $\{(a + \frac{\mathrm{e}^{-m\varepsilon}}{m} \sin ma, -\varepsilon - \frac{\mathrm{e}^{-m\varepsilon}}{m} \cos ma)\}$, respectively $\{(a + \frac{\mathrm{e}^{-mL}}{m} \sin ma, -L - \frac{\mathrm{e}^{-mL}}{m} \cos ma)\}$ (see figure 6). Observe also that the segment $\{(\pi/m, b) : -L < b < -\varepsilon\}$ is transformed by F into the segment $\{(\pi/m, b) : -L + \mathrm{e}^{-mL}/m < b < -\varepsilon + \mathrm{e}^{-m\varepsilon}/m\}$. This shows that there exist points A_0 in the interior of $\mathcal{U}_\varepsilon^L$ that are mapped in the open bounded set surrounded by $F(\partial\mathcal{U}_\varepsilon^L)$. From here we obtain at once that $\mathcal{W}_\varepsilon^L$ is contained in the compact set surrounded by $F(\partial\mathcal{U}_\varepsilon^L)$. Indeed, if a point A of the interior of $\mathcal{U}_\varepsilon^L$ is such that $F(A)$ does not belong to the open bounded set surrounded by $F(\partial\mathcal{U}_\varepsilon^L)$, then the curve $s \mapsto F(sA_0 + (1-s)A)$, $s \in [0, 1]$, will have to cross $F(\partial\mathcal{U}_\varepsilon^L)$ as it connects two points on different sides of this closed curve (see figure 7). However, an intersection point would be the image under F of two points in $\mathcal{U}$: one is on the boundary $\partial\mathcal{U}_\varepsilon^L$ and the other one is in the interior of $\mathcal{U}_\varepsilon^L$, lying on the segment $\{sA_0 + (1-s)A, s \in [0, 1]\}$. This contradicts the injectivity of the mapping F on $\mathcal{U}$. We have proved that $\mathcal{W}_\varepsilon^L$ is contained in the compact set surrounded by $F(\partial\mathcal{U}_\varepsilon^L)$.

The image of the compact set $\overline{\mathcal{U}_\varepsilon^L}$ (the closure of $\mathcal{U}_\varepsilon^L$) under F is a compact set. We claim now that it is precisely the closure of the bounded open set surrounded by $F(\partial\mathcal{U}_\varepsilon^L)$. If not, as $\partial\mathcal{U}_\varepsilon^L$ is mapped into $F(\partial\mathcal{U}_\varepsilon^L)$, there must be some point Z in the open bounded set enclosed by the curve $F(\partial\mathcal{U}_\varepsilon^L)$ that is not contained in $F(\overline{\mathcal{U}_\varepsilon^L})$. Since $F(\overline{\mathcal{U}_\varepsilon^L})$ is compact, there is an open ball B, centred at Z, which is in the open complementary of $F(\overline{\mathcal{U}_\varepsilon^L})$ and we take the ball with maximal radius. On the boundary of the ball B there is, by the maximality assumption, some point $Z_1 \in F(\overline{\mathcal{U}_\varepsilon^L})$. Since $F^{-1}(Z_1) = A_1 \in \mathcal{U}$ and F is a local diffeomorphism from $\mathcal{U}$ to $\mathcal{W}$, we can find a small neighbourhood of A_1 that is mapped by F onto a neighbourhood $\mathcal{N}$ of $Z_1 = F(A_1)$. However, then $\mathcal{N}$ would cover some points of B, contradicting the fact that by

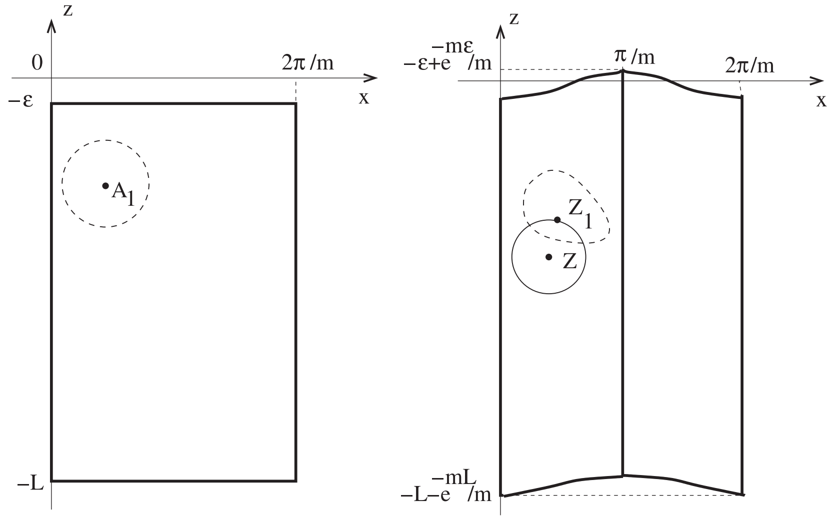

Figure 7.

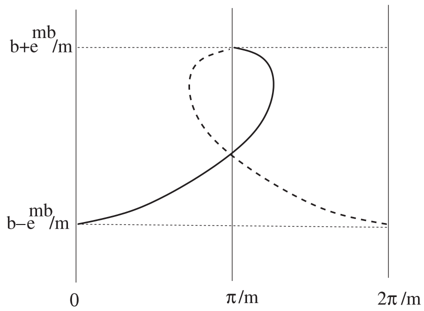

Figure 8.

construction B was entirely contained in $\mathbb{R}^2 - F(\overline{\mathcal{U}_\varepsilon})$. The obtained contradiction shows that $F(\overline{\mathcal{U}_\varepsilon})$ is exactly the compact set surrounded by $F(\partial\mathcal{U}_\varepsilon^L)$.

Step 1 is now justified by the combination of the proved facts. Indeed, if $b_0 \leqslant 0$ is given, we consider sets $\mathcal{U}_\varepsilon^L$ with $\varepsilon \downarrow |b_0|$ and $L \to \infty$ to infer that (2.2) is a global diffeomorphism from the domain below the horizontal line $\{(a, b_0) : a \in \mathbb{R}\}$ to the domain below the trochoid (or cycloid) obtained by setting $b = b_0$ in (2.2).

Remark. At this point we would like to comment on the importance of the assumption $b_0 \leqslant 0$: for fixed $b > 0$ and $t \geqslant 0$, the image under (2.1) of the horizontal line $\{(a, b) : a \in \mathbb{R}\}$ is a self-intersecting curve. Indeed, just like above, it is enough to analyse the case $a \in [0, 2\pi/m]$ with $t = 0$. If F_1 is the resulting vector function of a, note that its vertical component is increasing on $[0, \pi/m]$, while the horizontal component is increasing on $[0, \frac{1}{m}\arccos(-\mathrm{e}^{-mb})]$ and decreasing on $[\frac{1}{m}\arccos(-\mathrm{e}^{-mb}), \pi/m]$. Since $F_1(0) = (0, b - \mathrm{e}^{mb}/m)$, $F_1(\pi/m) = (\pi/m, b + \mathrm{e}^{mb}/m)$ and the points $F_1(\pi/m - a)$, $F_1(\pi/m + a)$ for $a \in [0, \pi/m]$ are symmetric with respect to the line $x = \pi/m$, the trochoid for $a \in [\pi/m, 2\pi/m]$ is obtained by reflecting the graph of F_1 in this line and we obtain the self-intersecting curve shown in figure 8. $\square$

Step 2. *The equation of the profile of the free surface is given by the image under* (2.1) *of the still water surface $z = b_0$, $b_0 \leqslant 0$:*

$$\begin{aligned} x &= a + \frac{e^{mb_0}}{m} \sin m\left(a + \sqrt{\frac{g}{m}}\, t\right) \\ z &= b_0 - \frac{e^{mb_0}}{m} \cos m\left(a + \sqrt{\frac{g}{m}}\, t\right). \end{aligned} \tag{2.3}$$

The boundary conditions (1.4) and (1.5) both hold.

The first part is contained in the considerations made throughout step 1. For the second assertion, observe that (1.4) expresses the fact that a particle on the free water surface will stay on it and this is already ensured. To prove the last part, note that the velocity of a particle with parameters (a, b) is obtained by differentiating (2.1) with respect to time,

$$u(t, x(t), y, z(t)) = \sqrt{\frac{g}{m}} \left(e^{mb} \cos m\left(a + \sqrt{\frac{g}{m}}\, t\right),\ 0,\ e^{mb} \sin m\left(a + \sqrt{\frac{g}{m}}\, t\right)\right) \tag{2.4}$$

of absolute value $\sqrt{\frac{g}{m}}\, e^{mb} \to 0$ as $z \to -\infty$ since $z \in [b - e^{mb}/m, b + e^{mb}/m]$.

Step 3. *The flow defined by* (2.1) *satisfies the equation of continuity.*

As already pointed out, the equation of continuity expresses incompressibility. It will be enough to show that the map

$$\begin{pmatrix} a + \dfrac{e^{mb}}{m} \sin ma \\ b - \dfrac{e^{mb}}{m} \cos ma \end{pmatrix} \mapsto \begin{pmatrix} a + \dfrac{e^{mb}}{m} \sin m\left(a + \sqrt{\dfrac{g}{m}}\, t\right) \\ b - \dfrac{e^{mb}}{m} \cos m\left(a + \sqrt{\dfrac{g}{m}}\, t\right) \end{pmatrix}$$

expressing the transformation of the initial state to the state at time $t > 0$, is area-preserving. To show that the Jacobian of this transformation is 1, note that the Jacobians of the transformations $\mathcal{T}_0$ and $\mathcal{T}$, defined by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a + \dfrac{e^{mb}}{m} \sin ma \\ b - \dfrac{e^{mb}}{m} \cos ma \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a + \dfrac{e^{mb}}{m} \sin m\left(a + \sqrt{\dfrac{g}{m}}\, t\right) \\ b - \dfrac{e^{mb}}{m} \cos m\left(a + \sqrt{\dfrac{g}{m}}\, t\right) \end{pmatrix}$$

respectively, are both time independent and equal to $(1 - e^{2mb})$. The map we are interested in is precisely $\mathcal{T} \circ \mathcal{T}_0^{-1}$ and its area-preserving property is now plain.

Step 4. *Associated with the flow* (2.1) *there is a unique hydrodynamical pressure P satisfying Euler's equations and the boundary condition* (1.3).

The acceleration of a particular fluid particle is obtained by differentiating (2.1) twice with respect to time,

$$\frac{Du}{Dt} = \left(-g e^{mb} \sin m\left(a + \sqrt{\frac{g}{m}}\, t\right),\ 0,\ g e^{mb} \cos m\left(a + \sqrt{\frac{g}{m}}\, t\right)\right)$$

so that the equations of motion (1.2) are

$$\begin{aligned} g e^{mb} \sin m\left(a + \sqrt{\frac{g}{m}}\, t\right) &= \frac{\partial}{\partial x}\left(\frac{P}{\rho} + gz\right) \\ -g e^{mb} \cos m\left(a + \sqrt{\frac{g}{m}}\, t\right) &= \frac{\partial}{\partial z}\left(\frac{P}{\rho} + gz\right) \end{aligned}$$

with the trivial y component eliminated. It appears to be more convenient to change variables from (x, z) to (a, b). Taking into account (2.1), this gives

$$\begin{aligned}\frac{\partial}{\partial a}\left(\frac{P}{\rho}+gz\right) &= g\mathrm{e}^{mb}\sin m\left(a+\sqrt{\frac{g}{m}}\,t\right)\\ \frac{\partial}{\partial b}\left(\frac{P}{\rho}+gz\right) &= g\mathrm{e}^{2mb}-g\mathrm{e}^{mb}\cos m\left(a+\sqrt{\frac{g}{m}}\,t\right).\end{aligned} \tag{2.5}$$

Expressing z from (2.1) and noting that the right-hand sides of both lines in (2.5) are the partial derivatives of $\left[\frac{g}{2m}\mathrm{e}^{2mb}-\frac{g}{m}\mathrm{e}^{mb}\cos m(a+\sqrt{\frac{g}{m}}t)\right]$ with respect to a, b, we obtain that for some constant c,

$$P = c-\rho g b+\frac{\rho g}{2m}\mathrm{e}^{2mb}. \tag{2.6}$$

This means that the hydrodynamical pressure is constant for all particles for which b is constant, irrespective of the value of a. In particular, the pressure has the same value for any given particle as it moves about. If we take $b = b_0 \leqslant 0$ as the still water surface, we obtain from (1.3) and (2.6) the explicit determination of the pressure at any particle during the flow (2.1),

$$P = P_0-\rho g(b-b_0)+\frac{\rho g}{2m}(\mathrm{e}^{2mb}-\mathrm{e}^{2mb_0}).$$

This was the last step in the proof of the fact that Gerstner's wave is an explicit solution to the governing equations for waves on deep water. We complete the examination of Gerstner's wave by analysing the *mean level* $z = k$, that is, the level with respect to which the same amount of water is elevated as depressed (see figure 9). Observe that, by definition, $\int(z-k)\,\mathrm{d}x = 0$ if taken over a wavelength of the surface profile. A change of variable transforms this relation to

$$\int_0^{2\pi/m}\left(b_0-k-\frac{\mathrm{e}^{mb_0}}{m}\cos\left[m\left(a+\sqrt{\frac{g}{m}}\,t\right)\right]\right)\left(1+\mathrm{e}^{mb_0}\cos\left[m\left(a+\sqrt{\frac{g}{m}}\,t\right)\right]\right)\mathrm{d}a = 0$$

which gives $k = b_0-\mathrm{e}^{2mb_0}/(2m)$. Therefore, the mean level is below the level of the still water by an amount $\mathrm{e}^{2mb_0}/(2m)$.

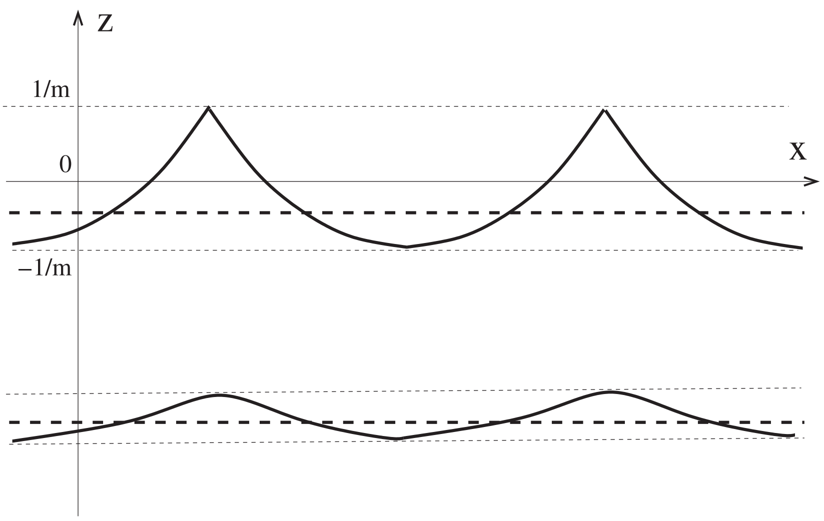

Figure 9.

Finally, let us point out that the motion of the water body induced by the passage of Gerstner's wave is rotational: the vorticity curl $u = \nabla \wedge u$ (measuring the local spin or rotation of a fluid element) is non-zero. Indeed, using the parametric representation of the location of the water particles and of the velocity, we obtain

$$\begin{aligned} u_1\,\mathrm{d}x + u_3\,\mathrm{d}z &= \left(\sqrt{\frac{g}{m}}\,\mathrm{e}^{2mb} + \sqrt{\frac{g}{m}}\,\mathrm{e}^{mb}\cos m\left(a + \sqrt{\frac{g}{m}}\,t\right)\right)\mathrm{d}a \\ &\quad + \sqrt{\frac{g}{m}}\,\mathrm{e}^{mb}\sin m\left(a + \sqrt{\frac{g}{m}}\,t\right)\mathrm{d}b \\ &= \mathrm{d}\left(\frac{1}{m}\sqrt{\frac{g}{m}}\,\mathrm{e}^{mb}\sin m\left(a + \sqrt{\frac{g}{m}}\,t\right)\right) + \sqrt{\frac{g}{m}}\,\mathrm{e}^{2mb}\,\mathrm{d}a. \end{aligned}$$

Since this is not an exact differential we see that the motion is rotational. To be more precise, let us fix an instant t and compute the value of the vorticity at a point (x_0, z_0) with parameters (a_0, b_0), located in the interior of the fluid domain (the vorticity being continuous, if we obtain its expression in the interior, we get it by a limiting process also for the boundary of the fluid domain). Let C_r^* be the image of the circle C_r of radius $r > 0$, centred at (a_0, b_0) and oriented clockwise, under the diffeomorphism (2.1). The diffeomorphism being orientation-preserving (its Jacobian is positive), C_r^* will be a smooth closed curve, lying in a vertical plane and oriented there in such a way that it leaves (x_0, z_0) always to its left by surrounding it. From Stokes's theorem we deduce that

$$n \cdot \operatorname{curl} u(x_0, z_0) = \lim_{r\to 0} \frac{1}{\mu_r} \int_{C_r^*} u \cdot \mathrm{d}l$$

where μ_r is the area of the surface determined by C_r^* in the (x, z)-plane, and $n = (0, -1, 0)$ is the normal. Changing variables by passing to the parameters (a, b), we have

$$\int_{C_r^*} u \cdot \mathrm{d}l = \int_{C_r} \mathrm{d}\left(\frac{1}{m}\sqrt{\frac{g}{m}}\,\mathrm{e}^{mb}\sin m\left(a + \sqrt{\frac{g}{m}}\,t\right)\right) + \sqrt{\frac{g}{m}}\,\mathrm{e}^{2mb}\,\mathrm{d}a.$$

Using the annihilation of an exact differential along a closed curve, we obtain

$$\int_{C_r^*} u \cdot \mathrm{d}l = \sqrt{\frac{g}{m}} \int_{C_r} \mathrm{e}^{2mb}\,\mathrm{d}a.$$

This can be expressed conveniently in polar coordinates $b = b_0 + r\sin\theta$, $a = a_0 + r\cos\theta$,

$$\int_{C_r} \mathrm{e}^{2mb}\,\mathrm{d}a = -r\mathrm{e}^{2mb_0} \int_0^{2\pi} \mathrm{e}^{2mr\sin\theta}\sin\theta\,\mathrm{d}\theta = -2mr^2\mathrm{e}^{2mb_0} \int_0^{2\pi} \mathrm{e}^{2mr\sin\theta}\cos^2\theta\,\mathrm{d}\theta$$

after integration by parts. We obtain that

$$n \cdot \operatorname{curl} u(x_0, z_0) = \lim_{r\to 0} \frac{-2\sqrt{gm}\,\mathrm{e}^{2mb_0}r^2}{\mu_r} \int_0^{2\pi} \mathrm{e}^{2mr\sin\theta}\cos^2\theta\,\mathrm{d}\theta.$$

According to multivariable calculus, the parametrizing diffeomorphism changes the area infinitesimally by a factor equal to the absolute value of the Jacobian, the latter being equal to $(1 - \mathrm{e}^{2mb})$ at a point (a, b). In particular, $\lim_{r\to 0} \frac{\mu_r}{\pi r^2} = 1 - \mathrm{e}^{2mb_0}$, so that

$$n \cdot \operatorname{curl} u(x_0, z_0) = -\frac{2\sqrt{gm}\,\mathrm{e}^{2mb_0}}{1 - \mathrm{e}^{2mb_0}}$$

as $\lim_{r\to 0} \int_0^{2\pi} \mathrm{e}^{2mr\sin\theta}\cos^2\theta\,\mathrm{d}\theta = \int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \pi$. Since we deal with a two-dimensional flow, one can easily see that curl $u = (0, \omega^*, 0)$ at any point. We infer that

$$\operatorname{curl} u = \left(0, \frac{2\sqrt{gm}\,\mathrm{e}^{2mb_0}}{1 - \mathrm{e}^{2mb_0}}, 0\right) = -\frac{2\sqrt{gm}\,\mathrm{e}^{2mb_0}}{1 - \mathrm{e}^{2mb_0}} n$$

at a point parametrized by (a_0, b_0). The negative sign indicates that the vorticity is in the opposite sense to the revolution of the particles in their circular orbits. Also, note that the vorticity decreases rapidly as we descend into the water.

3. Discussion

Let us emphasize some properties of Gerstner's wave and interpret them in the context of a mathematical understanding of the huge complexity of water waves—they confirm predictions of the linear theory[3].

The free water surface is generally a smooth trochoid with the extreme form of a cycloid with cusps upwards. These profiles are of wavelength $\lambda = 2\pi/m$ and of period $T = 2\pi/\sqrt{gm}$, as one can easily recognize from the parametric representation (2.3) of the surface wave: T is the time required for two successive crests to pass a fixed point in space while the horizontal distance between two successive crests is λ. Gerstner's wave is a symmetric wave whose profile—a trochoid or a cycloid—rises and falls exactly once per wavelength. The surface wave is expressed mathematically in the form

$$h(t, x) = h\left(x + \sqrt{\frac{g}{m}}\, t\right)$$

the periodic function h being the wave profile. That is, Gerstner's wave is a periodic progressive wave propagating at constant speed without change in shape, with a profile portraying accurately gravity waves that are observed in nature. The speed of propagation of Gerstner's wave is $c = \lambda/T$ or

$$c = \sqrt{\frac{g\lambda}{2\pi}}.$$

The previous relation is the dispersion relation for gravity waves on deep water. It shows that waves of different lengths travel at different speeds so that a group of waves of different lengths starting together would spread out. The dispersion relation is obtained (cf [3]) within the framework of the formal linear approximation to the governing equations, but in the case of Gerstner's wave the relation is derived rigorously as a byproduct of (2.1). This agreement supports the contention that the formal linear theory gives a fair representation of the propagation of water waves.

There is another fact pertinent to the present discussion. From the analysis in section 2 it is clear that any water particle describes a circle as Gerstner's wave passes, the radius of the circle getting smaller with increasing depth—a pattern convincingly verified by actual photographs (see [5]). Moreover, a glance at (2.4) shows that the speed $|\boldsymbol{u}|$ of a particle at depth larger than half of the wavelength, $b = b_0 - \lambda/2$, is smaller than the speed at the surface by a factor of at least $\mathrm{e}^{\pi} > 23$. Indeed, by (2.1), the square value of the speed of a particle parametrized by (a, b) is

$$\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2 = \frac{g}{m}\mathrm{e}^{2mb}.$$

The same conclusion holds for the size of the circular orbits of the particles as we can easily infer from (2.1): the circular path of a particle parametrized by (a, b) is of radius e^{mb}/m.

[3] This is especially interesting because Gerstner's wave is a solution to the nonlinear governing equations. The effects of nonlinearity are recognizable in that they distort its shape away from the sinusoidal waveforms $z = A\sin(lx - \omega t)$ encountered in linear water wave theory (see [2]), making crests narrower (up to the occurrence of peaks) and troughs flatter.

Therefore, at a depth larger than half the wavelength, the motion is less than 5% of its surface value. Gerstner's wave can thus be used to support the conclusion reached by formal linear approximation (see [2, 3]) that at depths below half-wavelength there is practically no motion.

We conclude that, in spite of its special character, Gerstner's wave presents features of general interest in that it assesses the value of basic conclusions about deep water waves, conclusions which, despite being obtained by a formal linearization approach (for which a rigorous justification is not available), are in good agreement with experiments. In addition, there is one aspect to which attention should particularly be drawn: the particular profile with upward cusps. It is a (perhaps perplexing) peculiarity to have an exact solution with peaks. Moreover, this solution describes the observed symmetrical peaking of the crests of water waves—according to [7] the occurrence of the intriguing peaking phenomenon in water waves is lost in the classical approximations to the governing equations.

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